Direct and inverse scattering on noncompact star-type quantum graph with Bessel singularity

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Abstract. In this paper we study the noncompact star-type graph with perturbed radial Schrodinger equation on each ray and the matching conditions of some special form at the vertex. The results include the uniqueness theorem and constructive procedure for solution of the inverse scattering problem.

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1 Introduction

Transport, spectral and scattering problems for differential operators on graphs appear frequently in mathematics, natural sciences and engineering [1], [2], [3], [4], [5], [6] and are in the focus of intensive investigations [7],[8], [9], [10], [11], [12], [13],[14], [15], [16].

In this paper we study the noncompact star-type graph with perturbed radial Schrodinger equation on each ray. Due to its physical importance this equation has obtained much attention and we refer for example to [17], [18], [19], [20], [21], [22] and the references therein. Our studies dealing with the equation on a metric graph can be treated as some continuation of the works [23], [24] (see also the references therein), where the important class of the Sturm–Liouville operators having Bessel-type singularity inside the interval was investigated. Matching conditions at the vertex arise as a generalization of matching conditions introduced in the works above under some additional conditions on behavior of the potential near the singularity. We call them the *matching conditions of analytical type* . One can notice that even for Sturm–Liouville expressions with real-valued potentials on finite interval such matching conditions can generate nonself-adjoint operators. For this reason we do not restrict our considerations to the case of real-valued potentials and real coefficients in matching conditions.

Our considerations follow in general the scheme recently developed for the investigation of inverse spectral problems on quantum graphs. The key point is the solution of "partial inverse problem" consisting in recovering the potential on one ray from the certain part of scattering data. Although in technical sense such problem looks similar to the classical inverse scattering problem for radial Schrodinger equation on the half-line, our approach differs from the approach used in the previous works [19]. Instead of classical Marchenko method we use a certain version of contour integral method based in part on the approach presented in [25] and in part on the ideas of spectral mappings method [26]. This allows us to provide the constructive procedure for solution the inverse problem based a certain linear equation with less restrictive requirements on the behavior of the potential at infinity; moreover, contrary to [19], we are able to prove the solvability of this equation. More exactly, under some conditions of regularity and "genericity" (which are similar to "genericity" conditions introduced in [25] and assume the absence of spectral singularities and some kind of regularity for small values of spectral parameter) our method works provided the potential satisfies at infinity only Marchenko decay condition.

Main results of the paper are contained in Theorem 5.1 and include the uniqueness result and constructive procedure for solution of the inverse scattering problem.

2 Direct scattering on the semi-axis. Auxiliary facts and notations.

Consider the equation:

$$-y'' + \left(\frac{\nu_0}{x^2} + q(x)\right) y = \lambda y = \rho^2 y,\ x > 0 \tag{2.1}$$

with complex ν_0 and complex-valued q that will be called as a *potential*. Let $\nu_0 = \nu^2 - 1/4$. For definiteness we assume that $\mathrm{Re}\nu > 1/2$, $\nu \notin \mathbb{N}$ and q satisfies the conditions:

$$\int\limits_0^1 \left|x^{1-2\nu} q(x)\right| dx + \int\limits_1^\infty |xq(x)|\, dx < \infty. \tag{2.2}$$

Here and below we assume $z^\mu := \exp(\mu \log|z| + i\mu \arg z)$, $\arg z \in (-\pi, \pi]$. Together with (2.1) we consider the unperturbed equation:

$$-y'' + \frac{\nu_0}{x^2} y = \lambda y = \rho^2 y,\ x > 0 \tag{2.3}$$

Denote $\mu_1 = 1/2 - \nu$, $\mu_2 = 1/2 + \nu$. Let $C_j(x, \lambda)$, $j = 1, 2$ be the solutions of unperturbed equation (2.3) defined as:

$$C_j(x,\lambda) = x^{\mu_j} \sum_{k=0}^\infty c_{jk} \lambda^k x^{2k},\ c_{10}c_{20} = (2\nu)^{-1},$$

$$c_{jk} = (-1)^k c_{j0} \left(\prod_{s=1}^k ((2s+\mu_j)(2s+\mu_j-1) - \nu_0) \right)^{-1}.$$

$C_j(x, \lambda)$ are both entire with respect to λ.

Define $S_j(x, \lambda)$, $j = 1, 2$ as the solutions of (2.1) satisfying the integral equations:

$$S_j(x,\lambda) = C_j(x,\lambda) - \int\limits_0^x g(x,t,\lambda) q(t) S_j(t,\lambda) dt,$$

where $g(x, t, \lambda)$ is a Green function: $g(x, t, \lambda) = C_1(x, \lambda)C_2(t, \lambda) - C_2(x, \lambda)C_1(t, \lambda)$. Note that the equations are solvable for both $j = 1, 2$ provided the condition (2.2). We also consider the Jost solution $f(x, \rho)$ for (2.1) normalized with the asymptotics $f^{(\xi)}(x, \rho) = \mathrm{e}^{i\rho x}((i\rho)^\xi + o(1))$, $x \to \infty$, $\xi = 0, 1$. Such solution exists (and unique) under the condition $xq(x) \in L_1(1, \infty)$, analytic with respect to $\rho \in \Omega_+ := \{\rho : \mathrm{Im}\rho > 0\}$ and $\rho^{-\mu_1} f^{(\xi)}(x, \rho)$ is continuous in $\rho \in \overline{\Omega}_+$.

The following expansion plays an important role in our further considerations:

$$f(x,\rho) = b_1(\rho) S_1(x,\lambda) + b_2(\rho) S_2(x,\lambda),$$

where coefficients $b_j(\rho)$, $j = 1, 2$ are called the *Stokes multipliers*. The ratio $b_2(\rho)/b_1(\rho)$ can be easily shown to coincide with the Weyl function $m(\lambda)$, which determines uniquely the potential q. This result and constructive procedure for solving the corresponding inverse problem one can find, for instance, in [27].

The following properties of $b_j(\rho)$ arise from the properties of Jost solution $f(x, \rho)$ and the results of [27].

Lemma 2.1. For $\rho \to \infty$ the following asymptotics holds:

$$b_j(\rho) = \rho^{\mu_j} \left(b_j^\infty + O(\rho^{-1}) \right),$$

where the constants $b_j^\infty \neq 0$ do not depend on q.

For $\rho \to 0$ the following asymptotics holds:

$$b_j(\rho) = \rho^{\mu_1} \left(b_j^0 + o(1) \right)$$

with some constants b_j^0 (depending, in general, on q).

3 Direct scattering on the graph

We consider the noncompact star-type metric graph Γ consisting of finite number of rays $\{\mathcal{R}_k\}_{k=1}^p$ emanating from their common initial vertex. A function y on the ray $\mathcal{R}_k$ we consider as a function of local parameter $x \in [0, \infty)$, where $x = 0$ corresponds to the vertex. A function y on Γ we consider as a set of functions $\{y_k\}_{k=1}^p$ (where $y_k = y\,|_{\mathcal{R}_k}$ is considered as a function on $[0, \infty)$ as it was specified above).

On each ray $\mathcal{R}_k$, $k = \overline{1, p}$ we consider the differential equation

$$-y'' + \left(\frac{\nu_{k0}}{x^2} + q_k(x) \right) y = \lambda y = \rho^2 y. \tag{3.1}$$

We follow the terminology of previous section and use the same notations adding the number k of ray as a first index. In particular, we assume $\nu_{k0} = \nu_k^2 - 1/4$, $\mathrm{Re}\nu_k > 1/2$, $\nu_k \notin \mathbb{N}$ and $q_k(x)$ to satisfy (2.2) (with $\nu = \nu_k$). The complex numbers ν_{k0} may be different for different rays but the following restriction is assumed throughout the paper.

Condition ν. If $\nu_j \neq \nu_k$ then $\mathrm{Re}\nu_j \neq \mathrm{Re}\nu_k$.

Let some function y satisfy (3.1). Then the Wronskians $\langle S_{k1}, y\rangle$ and $\langle y, S_{k2}\rangle$ do not depend on x and we can introduce the following linear forms:

$$U_{k1}(y) := \sigma_k \langle y, S_{k2}\rangle,$$

$$U_{k2}(y) := \sigma_{k1}\langle y, S_{k2}\rangle + \sigma_{k2}\langle S_{k1}, y\rangle.$$

Everywhere in the sequel we assume $\sigma_k \neq 0, \sigma_{k2} \neq 0$, $k = \overline{1,p}$.

Now let $y = \{y_k\}_{k=1}^p$ be a function on Γ such that each of y_k satisfies (3.1). We define the *matching conditions* at the vertex of Γ as follows:

$$U_{j1}(y_j) = U_{k1}(y_k), j \neq k,\ \sum_{j=1}^{p} U_{j2}(y_j) = 0. \tag{3.2}$$

Matching conditions (3.2) are generalization of matching conditions introduced in [23] for Sturm-Liouville operators having singularities inside the interval. They also can be considered as some generalization of standard (i.e. continuity together with Kirchhoff condition) matching conditions for quantum graphs.

With each ray $\mathcal{R}_k$ we associate the *Weyl-type solution* $\psi_k(\rho)$ that we determine as a function on Γ $\psi_k(\rho) = \{\psi_{kj}(x,\rho)\}_{j=1}^p$ such that:

- each of ψ_{kj}, $j = \overline{1,p}$ solves the differential equation (3.1);
- $\psi_{kj}(x,\rho) = O\left(\mathrm{e}^{i\rho x}\right)$ as $x \to \infty$ for $j \neq k$;
- $\psi_{kk}(x,\rho) = \mathrm{e}^{-i\rho x}(1 + o(1))$ as $x \to \infty$;
- $\psi_k(\rho)$ satisfies the matching conditions (3.2).

Let k be arbitrary fixed. We look for ψ_k in the following form:

$$\psi_{kj}(x,\rho) = \gamma_{kj}(\rho) f_j(x,\rho), j \neq k,\ \psi_{kk}(x,\rho) = \gamma_{kk}(\rho) f_k(x,\rho) + \frac{2i\rho}{b_{k1}(\rho)} S_{k2}(x,\lambda). \tag{3.3}$$

It is clear that any function of the form (3.3) with arbitrary coefficients γ_{kj} satisfies all the conditions determining the Weyl-type solution except matching conditions (3.2). Substituting (3.3) into (3.2) and taking into account that $\langle f_j, S_{j2}\rangle = b_{j1}$, $\langle S_{j1}, f_j\rangle = b_{j2}$ we arrive at the following linear algebraic system:

$$\sigma_j b_{j1}\gamma_{kj} + \beta_k = 0,\ j = \overline{1,p},\ \sum_{j=1}^{p}(\sigma_{j1}b_{j1} + \sigma_{j2}b_{j2})\gamma_{kj} = -\sigma_{k2}\delta_k,\ \delta_k = \frac{2i\rho}{b_{k1}} \tag{3.4}$$

with respect to the values $\gamma_{kj}, j = \overline{1,p}$, β_k. Cramer rule being applied to (3.4) together with representation (3.3) shows that $\psi_k(\rho)$ exists (and is unique) for all $\rho \in \overline{\Omega}_+$ outside the zeros of the function $b_{k1}(\rho)\Delta(\rho)$, where

$$\Delta(\rho) = \sum_{s=1}^{p}(\sigma_{s1}b_{s1}(\rho) + \sigma_{s2}b_{s2}(\rho)) \prod_{j\neq s}\sigma_j b_{j1}(\rho) \tag{3.5}$$

is a determinant of the system (3.4). In the sequel we assume that the following "genericity" condition holds.

Condition G. All the functions $b_{k1}(\rho)\Delta(\rho)$, $k = \overline{1,p}$ have no real zeros (with possible exception of $\rho = 0$) and no multiple zeros.

Let $\{1,...,p\} = \bigcup\limits_{\xi=1}^{m} I_\xi$, where the pairwise disjoint subsets I_ξ are such that for any ξ, $j,k \in I_\xi$ one has $\nu_j = \nu_k = \tau_\xi$ and $\mathrm{Re}\tau_1 > \ldots \mathrm{Re}\tau_m$. The following properties of characteristic function $\Delta(\rho)$ are the direct sequences of representation (3.5) and lemma 2.1.

Lemma 3.1. For $\rho \to \infty$ the following asymptotics holds:

$$\Delta(\rho) = \sum_{\xi=1}^{m} \rho^{\mu_1 + 2\tau_\xi} \left(a_\xi^\infty + O(\rho^{-1}) \right),$$

where

$$a_\xi^\infty = \sum_{s \in I_\xi} \sigma_{s2} b_{s2}^\infty \prod_{j \neq s} \sigma_j b_{j1}^\infty, \ \mu_1 := \sum_{j=1}^{p} \mu_{j1}.$$

For $\rho \to 0$ one has:

$$\Delta(\rho) = \rho^{\mu_1}(d^0 + o(1)), \ d^0 = \sum_{s=1}^{p} (\sigma_{s1} b_{s1}^0 + \sigma_{s2} b_{s2}^0) \prod_{j \neq s} \sigma_j b_{j1}^0.$$

It is important to notice that the coefficients a_ξ^∞ depend on the numbers ν_k, $k = \overline{1,p}$ and coefficients $\sigma_k, \sigma_{k1}, \sigma_{k2}$ in matching conditions (3.2) and do not depend on the potentials q_k. In what follows we assume that the following conditions of "regularity" and "genericity at 0" hold.

Condition R_∞. $a_1^\infty \neq 0$.

Condition R_0. $b_{j1}^0 \neq 0$, $j = \overline{1,p}$, $d^0 \neq 0$.

Our further considerations deal with $\psi_{kk}(x, \rho)$. Using (3.3) and the representations:

$$\gamma_{kk}(\rho) = \delta_k(\rho) \cdot \frac{\Delta_k(\rho)}{\Delta(\rho)}, \ \delta_k(\rho) = \frac{2i\rho}{b_{k1}(\rho)}, \ \Delta_k(\rho) = (\sigma_k)^2 \prod_{j \neq k} \sigma_j b_{j1}(\rho), \tag{3.6}$$

that can be obtained from (3.4) using the Cramer rule we deduce the following assertion.

Lemma 3.2. Under the Condition R_0 one has

$$\gamma_{kk}(\rho) = O\left(\rho^{1-2\mu_{k1}}\right)$$

as $\rho \to 0$. Moreover,

$$\psi_{kk}^{(\xi-1)}(x, \rho) = O\left(\rho^{\mu_{k2}}\right), \ \xi = 0, 1$$

as $\rho \to 0$ for any fixed $x > 0$.

It follows from lemmas 2.1, 3.1 that under the conditions G, R_0 and R_∞ the function $b_{k1}(\rho)\Delta(\rho)$ has at most finite set of simple zeros, $\psi_{kk}(x, \rho)$ has at these points either a simple pole or a removable singularity. Denote the set of poles of $\psi_{kk}(x, \rho)$ in Ω_+ as Z_k^+. It follows from representation (3.3) that for any $\rho_0 \in Z_k^+$ one has:

$$\text{res}_{\rho=\rho_0} \psi_{kk}(x, \rho) = \alpha_k(\rho_0) f_k(x, \rho_0) = \alpha_k(\rho_0) \exp(i\rho_0 x)(1 + o(1)), x \to \infty. \tag{3.7}$$

with some constant $\alpha_k(\rho_0)$. Indeed, (3.7) is obvious if ρ_0 is a zero of $\Delta(\rho)$; if ρ_0 is a zero of $b_{k1}(\rho)$ we take into account that $S_{k2}(x, \rho_0)$ is proportional to $f_k(x, \rho_0)$ and arrive again to (3.7) (let us recall that under the condition G $\Delta(\rho)$ and $b_{k1}(\rho)$ have no common zeros).

Further, for real nonzero ρ usual arguments yield the following representation:

$$\psi_{kk}(x, \rho) = f_k(x, -\rho) + r_k(\rho) f_k(x, \rho) = \exp(-i\rho x) + r_k(\rho) \exp(i\rho x) + o(1), x \to \infty. \tag{3.8}$$

We call r_k the *reflection coefficient associated with the ray* $\mathcal{R}_k$, the set $J_k := \{r_k, Z_k^+, \alpha_k(\rho), \rho \in Z_k^+\}$ is called the *scattering data associated with the ray* $\mathcal{R}_k$, the set $J = \{J_k\}_{k=1}^{p-1}$ is called the *scattering data for* Γ.

4 Inverse scattering. Partial problem.

This section is devoted to the following partial inverse problem.

Problem $IP(k)$. Given J_k, $k \in \{1, \ldots, p\}$ find $q_k(x), x > 0$.

Let L denotes the problem on Γ consisting of differential equations (3.1) on each ray $\mathcal{R}_j$, $j=\overline{1,p}$ and matching conditions (3.2). Together with L we consider the problem $\tilde{L}$ with equations of the same form (3.1) and matching conditions (3.2) but different potentials $\tilde{q}_j$, $j=\overline{1,p}$. We assume that coefficients in (3.2) are the same for L and $\tilde{L}$ and $\tilde{\nu}_{k0}=\nu_{k0}$. Moreover, we assume that the conditions G, R_0 and R_∞ hold for both L and $\tilde{L}$. We agree that if η denotes some object related to L then $\tilde{\eta}$ denotes an analogous object related to $\tilde{L}$ and $\hat{\eta}:=\eta-\tilde{\eta}$.

In what follows we shall also use the following notations. If A denotes some matrix then A_j denotes its j-th row. If f is some function holomorphic in deleted neighborhood of ρ_0 then $f_{\langle m\rangle}(\rho_0)$ denote the coefficients in the Laurent expansion:

$$f(\rho)=\sum_{m=-\infty}^{\infty}(\rho-\rho_0)^m f_{\langle m\rangle}(\rho_0).$$

If $f(\rho)$ is some function meromorphic outside the real axis then for real ρ we denote $f_\pm(\rho):=\lim_{\varepsilon\to+0} f(\rho\pm i\varepsilon)$.

From this point in this section we assume that k is arbitrary fixed. The key role in our further considerations is played by the *spectral mappings matrix* $P(x,\rho):=\Psi(x,\rho)\tilde{\Psi}^{-1}(x,\rho)$, where:

$$\Psi(x,\rho):=\begin{bmatrix}\psi_{kk}(x,\rho) & f_k(x,\rho)\\ \psi'_{kk}(x,\rho) & f'_k(x,\rho)\end{bmatrix},\ \rho\in\Omega_+,\ \Psi(x,\rho):=\Psi(x,-\rho),\ \rho\in\Omega_-:=\{\rho:\mathrm{Im}\rho<0\}. \tag{4.1}$$

Lemma 4.1. For each fixed $x>0$ $P(x,\rho)$ is bounded for $\rho\to0$ and $\rho\to\infty$. Moreover, for $\rho\to\infty$ one has $P_1(x,\rho)=I_1+O(\rho^{-1})$, where (and everywhere below) I denotes the identity matrix.

Proof. By virtue of symmetry it is sufficient to consider ρ from upper half-plane $\overline{\Omega}_+$. For nonzero $\rho\in\overline{\Omega}_+$ one has $\det\Psi=\det\tilde{\Psi}=2i\rho$ and:

$$2i\rho P_{\xi1}(x,\rho)=\psi_{kk}^{(\xi-1)}(x,\rho)\tilde{f}'_k(x,\rho)-f_k^{(\xi-1)}(x,\rho)\tilde{\psi}'_{kk}(x,\rho),$$

$$2i\rho P_{\xi2}(x,\rho)=f_k^{(\xi-1)}(x,\rho)\tilde{\psi}_{kk}(x,\rho)-\psi_{kk}^{(\xi-1)}(x,\rho)\tilde{f}_k(x,\rho).$$

Lemma 3.2 together with the estimate $f_k^{(\xi-1)}(x,\rho)=O\left(|\rho^{\mu_{k1}}|\right)$ provide the boundedness of $P(x,\rho)$ for $\rho\to0$. For $\rho\to\infty$ we use the asymptotics:

$$f_k^{(\xi)}(x,\rho)=(i\rho)^\xi\exp(i\rho x)[1],\ S_{k2}^{(\xi)}(x,\lambda)=\beta_{k0}\rho^{-\mu_{k2}}\left((-i\rho)^\xi\exp(-i\rho x)[1]+(i\rho)^\xi\gamma_{k0}\exp(i\rho x)[1]\right), \tag{4.2}$$

where $[1]:=1+O(\rho^{-1})$ and the constants β_{k0}, γ_{k0} depend only on ν_{k0} (and therefore are the same for L and $\tilde{L}$). In particular this yields the estimates:

$$\hat{f}_k^{(\xi)}(x,\rho)=O\left((i\rho)^{\xi-1}\exp(i\rho x)\right),\ \hat{S}_{k2}^{(\xi)}(x,\lambda)=O\left(\rho^{-\mu_{k2}+\xi-1}\exp(-i\rho x)\right). \tag{4.3}$$

Moreover, lemma 3.1 provides the estimates:

$$\hat{\Delta}(\rho)=O\left(\rho^{\mu_1+2\tau_1-1}\right),\ \Delta^{-1}(\rho)=O\left(\rho^{-\mu_1-2\tau_1}\right). \tag{4.4}$$

On the other hand one has the asymptotics (following directly from (3.5)):

$$\Delta_k(\rho)=\rho^{\mu_1-\mu_{k1}}(d_k^\infty+O(\rho^{-1})),\ d_k^\infty=\sigma_k^2\prod_{j\neq k}\sigma_j b_{j1}^\infty.$$

Together with lemma 2.1 and the estimates (4.4) this yields:

$$\gamma_{kk}(\rho)=O\left(\rho^{2\nu_{k1}-2\tau_1}\right)=O(1),\ \hat{\gamma}_{kk}(\rho)=O(\rho^{-1}\gamma_{kk}(\rho))=O(\rho^{-1}). \tag{4.5}$$

The asymptotics (4.2), (4.3), (4.5) provide the estimates:

$$\psi_{kk}^{(\xi)}(x,\rho)=O\left(\rho^\xi\exp(-i\rho x)\right),\ \hat{\psi}_{kk}^{(\xi)}(x,\rho)=O\left(\rho^{\xi-1}\exp(-i\rho x)\right). \tag{4.6}$$

Together with (4.2), (4.3) the estimates (4.6) yield the required boundedness of $P(x,\rho)$ for $\rho\to\infty$ and the estimate $P_{12}(x,\rho)=O(\rho^{-1})$. In order to obtain the estimate for $P_{11}(x,\rho)-1$ we write:

$$2i\rho P_{11}(x,\rho)=\psi_{kk}(x,\rho)\tilde{f}'_k(x,\rho)-f_k(x,\rho)\tilde{\psi}'_{kk}(x,\rho)=2i\rho+\hat{\psi}_{kk}(x,\rho)\tilde{f}'_k(x,\rho)-\hat{f}_k(x,\rho)\tilde{\psi}'_{kk}(x,\rho)$$

(where we take into account that $\langle\tilde{\psi}_{kk}, \tilde{f}_k\rangle = 2i\rho$) and use again (4.2), (4.3), (4.6). $\square$

It is clear that $P(x,\rho)$ is meromorphic in ρ outside the real axis and (under condition G) for all nonzero real ρ has the limits $P_\pm(x,\rho)$ which are continuous and bounded on $\mathbb{R}\setminus\{0\}$. All the possible poles of $P(x,\rho)$ are necessarily belong to the set $\check{Z}_k := Z_k \cup \tilde{Z}_k$, where $Z_k = \{\pm\rho, \rho \in Z_k^+\}$.

Lemma 4.2. Any $\rho_0 \in \check{Z}_k$ is either a simple pole or a removable singularity for $P(x,\rho)$. Moreover, the following representation holds:

$$P_{\langle -1\rangle}(x,\rho_0) = \Psi_{\langle 0\rangle}(x,\rho_0)\hat{v}(\rho_0)(\tilde{\Psi}^{-1})_{\langle 0\rangle}(x,\rho_0),$$

where $v(\rho_0) := 0$ if $\rho_0 \notin Z_k$ and

$$v(\rho_0) = \begin{bmatrix} 0 & \alpha_k(\rho_0) \\ 0 & 0 \end{bmatrix}$$

if $\rho_0 \in Z_k$. Here $\alpha_k(\rho_0), \rho_0 \in Z_k^+$ are the constants from (3.7) and $\alpha_k(\rho_0) := -\alpha_k(-\rho_0)$ for $\rho_0 \in Z_k \cap \Omega_-$.

Proof. It follows from (3.7) that $\rho_0 \in \check{Z}_k$ is (at most) a simple pole for $\Psi(x,\rho)$ and the following relation holds:

$$\Psi_{\langle -1\rangle}(x,\rho_0) = \Psi_{\langle 0\rangle}(x,\rho_0)v(\rho_0). \tag{4.7}$$

Moreover, since $\det\Psi = \det\tilde{\Psi} = \pm 2i\rho$ for $\pm\rho \in \Omega_+$, the matrix $\tilde{\Psi}^{-1}(x,\rho)$ has (at most) a simple pole in ρ_0 and from $\tilde{\Psi}\tilde{\Psi}^{-1} = I$ one can easily obtain:

$$(\tilde{\Psi}^{-1})_{\langle -1\rangle}(x,\rho_0) = -\tilde{v}(\rho_0)(\tilde{\Psi}^{-1})_{\langle 0\rangle}(x,\rho_0). \tag{4.8}$$

Thus, $P(x,\rho)$ has in ρ_0 a pole of multiplicity 2 or less. But $P_{\langle -2\rangle} = \Psi_{\langle -1\rangle}(\tilde{\Psi}^{-1})_{\langle -1\rangle} = -\Psi_{\langle 0\rangle}v\tilde{v}(\tilde{\Psi}^{-1})_{\langle 0\rangle}$. We notice that because of special structure of matrices $v(\rho_0)$ and $\tilde{v}(\rho_0)$ we have $v(\rho_0)\tilde{v}(\rho_0) = 0$ and therefore $P_{\langle -2\rangle}(x,\rho_0) = 0$. Now we calculate $P_{\langle -1\rangle} = \Psi_{\langle -1\rangle}(\tilde{\Psi}^{-1})_{\langle 0\rangle} + \Psi_{\langle 0\rangle}(\tilde{\Psi}^{-1})_{\langle -1\rangle}$ and using (4.7) and (4.8) obtain the required representation. $\square$

Our first result in this section is the following uniqueness theorem.

Theorem 4.1. $J_k = \tilde{J}_k$ implies $q_k(x) = \tilde{q}_k(x)$ for a.e. $x > 0$, i.e. specification of the scattering data associated with some ray $\mathcal{R}_k$ uniquely determines the potential q_k on this ray.

Proof. First we notice that (3.8) yields the following relation for the matrices $\Psi_\pm(x,\rho)$:

$$\Psi_+(x,\rho) = \Psi_-(x,\rho)v(\rho),\ \ \rho \in \mathbb{R}\setminus\{0\} \tag{4.9}$$

with "jump matrix"

$$v(\rho) = \begin{bmatrix} r_k(\rho) & 1 \\ 1 - r_k(\rho)r_k(-\rho) & -r_k(-\rho) \end{bmatrix}. \tag{4.10}$$

From (4.10) one can notice that $\tilde{r}_k = r_k$ implies $\tilde{v}(\rho) = v(\rho)$ for all $\rho \in \mathbb{R}\setminus\{0\}$ and therefore $P_+(x,\rho) = P_-(x,\rho)$ for each fixed $x > 0$ and all nonzero real ρ. Thus, if $\tilde{J}_k = J_k$ the matrix $P(x,\rho)$ is meromorphic in $\mathbb{C}\setminus\{0\}$ with possible poles in points of the set $\check{Z}_k$. Moreover, $\tilde{J}_k = J_k$ implies $\check{Z}_k = \tilde{Z}_k = Z_k$ and $\tilde{v}(\rho_0) = v(\rho_0)$ for any $\rho_0 \in Z_k$. Then, by virtue of lemma 4.2 we have $P_{\langle -1\rangle}(x,\rho_0) = 0$, i.e. $P(x,\rho)$ has actually removable singularities in all $\rho_0 \in Z_k$. Thus, $P(x,\rho)$ is holomorphic in $\mathbb{C}\setminus\{0\}$ and by virtue of lemma 4.1 bounded for $\rho \to 0$ and $\rho \to \infty$. Moreover, by virtue of lemma 4.1 $P_1(x,\rho) - I_1$ vanishes at infinity. This means that $P_1(x,\rho) - I_1$ is actually equal to 0 identically. Thus, we have $P_1(x,\rho) \equiv I_1$ that yields $f_k(x,\rho) \equiv \tilde{f}_k(x,\rho)$ and subsequently $q_k(x) = \tilde{q}_k(x)$ for a.e. $x > 0$. $\square$

Our next goal is a constructive procedure for solving the problem $IP(k)$. From this point we assume that the numbers ν_{j0}, $j = \overline{1,p}$ and coefficients of the forms U_{j1}, U_{j2} in matching conditions (3.2) are known. Also we suppose that $\tilde{L}$ is chosen a priori "model" (or "reference") problem with known potentials $q_j, j = \overline{1,p}$. Below we reduce the problem $IP(k)$ to a certain linear equation ("main equation") in some Banach space and prove the unique solvability of this equation.

We consider again the spectral mappings matrix $P(x,\rho)$. By virtue of lemma 4.1 one has $\int_{|\mu|=R}(\mu - \rho)^{-1}d\mu(P_1(x,\mu) - I_1) \to 0$ as $R \to \infty$ that yields the following basic relation:

$$P_1(x,\rho) - I_1 = \sum_{\mu\in\check{Z}_k}(\rho-\mu)^{-1}P_{1,\langle -1\rangle}(x,\mu) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu}{\mu-\rho}\left(P_{1,+}(x,\mu) - P_{1,-}(x,\mu)\right) \tag{4.11}$$

with arbitrary $\rho \in \mathbb{C} \setminus (\mathbb{R} \cup \check{Z}_k)$.

Let us introduce the matrix $V(x,\rho) := P_-^{-1}(x,\rho)P_+(x,\rho) = \tilde{\Psi}_-(x,\rho)v(\rho)\tilde{\Psi}_+^{-1}(x,\rho)$, $\rho \in \mathbb{R} \setminus \{0\}$. We notice that V is uniquely determined by $\tilde{L}$ and scattering data J_k. We also note that in view of lemma 4.1 and $\det P = 1$, $V(x,\rho)$ is continuous and bounded with respect to $\rho \in \mathbb{R} \setminus \{0\}$ and any fixed $x > 0$.

We define

$$d_1(x,\rho,\mu_0) = \left[(\rho-\mu)^{-1}\Psi^{-1}(x,\mu)\right]_{\langle 0\rangle}\Big|_{\mu=\mu_0}, \ \rho \in \mathbb{C} \setminus \check{Z}_k, \mu_0 \in \check{Z}_k$$

$$d_2(x,\rho_0,\mu) = \left[(\rho-\mu)^{-1}\Psi(x,\rho)\right]_{\langle 0\rangle}\Big|_{\rho=\rho_0}, \ \mu \in \mathbb{C} \setminus \check{Z}_k, \rho_0 \in \check{Z}_k$$

$$D(x,\rho,\mu) := (\rho-\mu)^{-1}(\Psi^{-1})(x,\mu)\Psi(x,\rho), \ \rho \neq \mu, \rho,\mu \in \mathbb{C} \setminus \check{Z}_k,$$

$$D(x,\rho,\mu_0) := [D(x,\rho,\mu)]_{\langle 0\rangle}\Big|_{\mu=\mu_0}, \ \rho \in \mathbb{C} \setminus \check{Z}_k, \mu_0 \in \check{Z}_k$$

$$D(x,\rho_0,\mu) := [D(x,\rho_0,\mu)]_{\langle 0\rangle}\Big|_{\rho=\rho_0}, \ \mu \in \mathbb{C} \setminus \check{Z}_k, \rho_0 \in \check{Z}_k$$

$$D(x,\rho_0,\mu_0) := [D(x,\rho,\mu_0)]_{\langle 0\rangle}\Big|_{\rho=\rho_0} = [D(x,\rho_0,\mu)]_{\langle 0\rangle}\Big|_{\mu=\mu_0}, \ \rho_0 \in \check{Z}_k, \ \mu_0 \in \check{Z}_k$$

and $\tilde{d}_j(x,\rho,\mu)$, $j = 1,2$, $\tilde{D}(x,\rho,\mu)$ by similar formulae, where $\Psi(x,\rho)$ is replaced with $\tilde{\Psi}(x,\rho)$. We also define:

$$\tilde{A}(x,\rho,\mu) := \tilde{D}(x,\rho,\mu)\hat{v}(\rho), \ A(x,\rho,\mu) := -D(x,\rho,\mu)\hat{v}(\rho), \ \rho \in \check{Z}_k, \ \mu \in \check{Z}_k.$$

Lemma 4.3. For $\mu \in \check{Z}_k$, $\rho \notin \check{Z}_k$ one has:

$$(\rho-\mu)^{-1}P_{\langle -1\rangle}(x,\mu) = \Psi_{\langle 0\rangle}(x,\mu)\hat{v}(\mu)\tilde{d}_1(x,\rho,\mu),$$

$$(\rho-\mu)^{-1}P_{\langle -1\rangle}(x,\mu)\tilde{\Psi}(x,\rho) = \Psi_{\langle 0\rangle}(x,\mu)\hat{v}(\mu)\tilde{D}(x,\rho,\mu).$$

For $\xi \in \check{Z}_k$, $\rho,\mu \notin \check{Z}_k$

$$(\rho-\xi)^{-1}(\xi-\mu)^{-1}\Psi^{-1}(x,\mu)P_{\langle -1\rangle}(x,\xi)\tilde{\Psi}(x,\rho) = D(x,\xi,\mu)\hat{v}(\xi)\tilde{D}(x,\rho,\xi).$$

Proof. The relations can be easily verified by direct calculation. Consider, for instance,

$$\tilde{d}_1(x,\rho,\mu_0) = \left[(\rho-\mu)^{-1}\tilde{\Psi}^{-1}(x,\mu)\right]_{\langle 0\rangle}\Big|_{\mu=\mu_0} =$$

$$(\rho-\mu_0)^{-1}(\tilde{\Psi}^{-1})_{\langle 0\rangle}(x,\mu_0) + \left[(\rho-\mu)^{-1}\right]_{\langle 1\rangle}\Big|_{\mu=\mu_0} \cdot (\tilde{\Psi}^{-1})_{\langle -1\rangle}(x,\mu_0).$$

Multiplying this by $\Psi_{\langle 0\rangle}(x,\mu_0)\hat{v}(\mu_0)$, then using lemma 4.2, relation (4.8) and taking into account that $\hat{v}(\mu_0)\tilde{v}(\mu_0) = 0$ we obtain the first of the required relations. The others can be obtained in a similar way.□

The assertion below follows directly from definition of $d_j(x,\rho,\mu)$.

Lemma 4.4. For any fixed $\mu \in \check{Z}_k$ $d_1(x,\rho,\mu)$ is holomorphic in $\rho \in \mathbb{C} \setminus \check{Z}_k$. In particular for real ρ one has $d_1(x,\rho+i0,\mu) = d_1(x,\rho-i0,\mu) = d_1(x,\rho,\mu)$, moreover, $d_1(x,\cdot,\mu) \in L_2\left(\mathbb{R},\mathbb{C}^2\right)$. Analogously, for any fixed $\rho \in \check{Z}_k$ $d_2(x,\rho,\mu)$ is holomorphic in $\mu \in \mathbb{C} \setminus \check{Z}_k$, $d_2(x,\rho,\cdot) \in L_2\left(\mathbb{R},\mathbb{C}^2\right)$. Here and below elements of $\mathbb{C}^2$ are considered as row-vectors.

In our further considerations we will also use the Cauchy operators:

$$(Cf)(\rho) := \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu}{\mu-\rho}f(\mu), \ \rho \in \mathbb{C} \setminus \mathbb{R},$$

$$(C_\pm f)(\rho) := (Cf)_\pm(\rho), \ \rho \in \mathbb{R}$$

and the Plemelj-Sokhotskii formula $C_\pm = \mathcal{P} \pm E/2$. Here (and everywhere below) E denotes the identical operator and

$$(\mathcal{P}f)(\rho) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu}{\mu-\rho}f(\mu), \ \rho \in \mathbb{R}$$

where a principal value of integral is assumed. We recall that $C_\pm$ and $\mathcal{P}$ are bounded operators in $L_2(\mathbb{B},\mathcal{M})$ (for any finite-dimensional space $\mathcal{M}$) and the following relation holds for two arbitrary matrices F_1, F_2 of suitable orders with quadratically integrable elements:

$$\int_{-\infty}^{\infty} d\mu (C_\pm F_1)(\mu) F_2(\mu) = -\int_{-\infty}^{\infty} d\mu F_1(\mu)(C_\mp F_2)(\mu).$$

Define $\Phi(x,\rho) := P_{1,+}(x,\rho) - P_{1,-}(x,\rho)$ for real ρ and $\Phi(x,\rho) := \Psi_{1,\langle 0\rangle}(x,\rho)\hat{v}(\rho)$ for $\rho \in \check{Z}_k$. One can easily notice that by virtue of lemma 4.1 $\Phi(x,\cdot)|_{\mathbb{R}} \in L_2\left(\mathbb{R},\mathbb{C}^2\right)$. Now we return to the relation (4.11) and using the notations introduced above and lemma 4.3 rewrite it as follows:

$$P_1(x,\rho) - I_1 = \sum_{\mu\in\check{Z}_k} \Phi(x,\mu)\tilde{d}_1(x,\rho,\mu) + (C\Phi(x,\cdot))(\rho),\ \ \rho\in\mathbb{C}\setminus(\mathbb{R}\cup\check{Z}_k). \tag{4.12}$$

Taking in (4.12) the limits as $\pm\mathrm{Im}\rho \to 0$ and substituting them into the jump relation $P_{1,+} - P_{1,-}V = 0$ we arrive at:

$$\Phi(x,\rho)\left(\frac{1}{2}I + \frac{1}{2}V(x,\rho)\right) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\mu}{\mu-\rho}\Phi(x,\mu)(I - V(x,\rho)) +$$

$$\sum_{\mu\in\check{Z}_k} \Phi(x,\mu)\tilde{d}_1(x,\rho,\mu)(I - V(x,\rho)) + I_1 - V_1(x,\rho) = 0,\ \ \rho\in\mathbb{R}\setminus\{0\}, \tag{4.13}$$

where a principal value of integral is assumed. In the particular case when $\check{Z}_k = \emptyset$ the relation (4.13) can be considered for each fixed $x > 0$ as a linear equation with respect to $\Phi(x,\cdot)$. In general case (4.13) should be completed with some relations at the points $\rho \in \check{Z}_k$. We proceed as follows. Multiplying (4.12) by $\tilde{\Psi}(x,\rho)$ and using lemma 4.3 we obtain:

$$\Psi_1(x,\rho) - \tilde{\Psi}_1(x,\rho) = \sum_{\mu\in\check{Z}_k} \Psi_{1,\langle 0\rangle}(x,\mu)\hat{v}(\mu)\tilde{D}(x,\rho,\mu) + (C\Phi(x))(\rho)\tilde{\Psi}(x,\rho),\ \ \rho\in\mathbb{C}\setminus(\mathbb{R}\cup\check{Z}_k).$$

Now we take an arbitrary $\rho_0 \in \check{Z}_k$ and multiply the relation above by $\hat{v}(\rho_0)$. Thus we get:

$$\Psi_1(x,\rho)\hat{v}(\rho_0) - \tilde{\Psi}_1(x,\rho)\hat{v}(\rho_0) =$$

$$\sum_{\mu\in\check{Z}_k} \Psi_{1,\langle 0\rangle}(x,\mu)\hat{v}(\mu)\tilde{D}(x,\rho,\mu)\hat{v}(\rho_0) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu}{\mu-\rho}\Phi(x,\mu)\tilde{\Psi}(x,\rho)\hat{v}(\rho_0). \tag{4.14}$$

Consider again the relation (4.14). Taking the coefficient $[\ldots]_{\langle 0\rangle}\big|_{\rho=\rho_0}$ in Laurent series of its both sides and using lemma 4.3 we arrive at:

$$\Phi(x,\rho_0) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mu\Phi(x,\mu)\tilde{d}_2(x,\rho_0,\mu)\hat{v}(\rho_0) - \sum_{\mu\in\check{Z}_k}\Phi(x,\mu)\tilde{A}(x,\rho_0,\mu) = \tilde{\Psi}_{1,\langle 0\rangle}(x,\rho_0)\hat{v}(\rho_0), \rho_0\in\check{Z}_k, \tag{4.15}$$

where $\rho_0 \in \check{Z}_k$ is arbitrary. Together with (4.13) (4.15) forms the complete linear system with respect to $\Phi(x,\rho)$, $\rho\in\mathbb{R}\cup\check{Z}_k$. We can summarize our considerations as follows.

Theorem 4.2. Let $\tilde{d}_j(x,\rho,\mu)$ and $\tilde{A}(x,\rho,\mu)$ constructed as described above by given scattering data J_k and given model problem $\tilde{L}$. Then

$$\Phi(x,\rho) = \begin{cases} P_{1,+}(x,\rho) - P_{1,-}(x,\rho),\ \ \rho\in\mathbb{R}, \\ \Psi_{1,\langle 0\rangle}(x,\rho)\hat{v}(\rho), \rho\in\check{Z}_k \end{cases}$$

for each fixed $x > 0$ solves the linear system (4.13), (4.15).

In order to complete our procedure of solving the problem $IP(k)$ we are to show the unique solvability of the specified linear system. First we rewrite the system as a linear equation in Banach space $\mathcal{H} := \mathcal{H}_r \oplus \mathcal{H}_d$,

where $\mathcal{H}_r = L_2(\mathbb{R},\mathbb{C}^2)$, $\mathcal{H}_d = (\mathbb{C}^2)^{\check{Z}_k}$. As we already mentioned above for each fixed $x > 0$ $\Phi(x,\cdot)$ can be considered as an element of $\mathcal{H}$. Thus, the system (4.13), (4.15) can be written in the form $\mathcal{A}\Phi = G$, where

$$G(\rho) := \begin{cases} V_1(x,\rho) - I_1, \ \rho \in \mathbb{R}, \\ \tilde{\Psi}_{1,\langle 0\rangle}(x,\rho)\hat{v}(\rho), \rho \in \check{Z}_k, \end{cases}$$

while $\mathcal{A}$ is a linear bounded operator in $\mathcal{H}$ defined by the following operator matrix:

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{rr} & \mathcal{A}_{rd} \\ \mathcal{A}_{dr} & \mathcal{A}_{dd} \end{bmatrix},$$

$$\mathcal{A}_{rr}\varphi = C_+\varphi - (C_-\varphi)V, \ (\mathcal{A}_{rd}\varphi)(\rho) = \sum_{\mu\in\check{Z}_k} \varphi(\mu)\tilde{d}_1(\rho,\mu)(I - V(\rho)), \ \rho \in \mathbb{R},$$

$$(\mathcal{A}_{dr}\varphi)(\rho) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \varphi(\xi)\tilde{d}_2(\rho,\xi)\hat{v}(\rho)d\xi, \ \rho \in \check{Z}_k, \ (\mathcal{A}_{dd}\varphi)(\rho) = \varphi(\rho) - \sum_{\mu\in\check{Z}_k} \varphi(\mu)\tilde{A}(x,\rho,\mu), \ \rho \in \check{Z}_k.$$

Here and below we assume that parameter $x > 0$ is (arbitrary) fixed and for the sake of brevity we omit it in all the argument's lists. Boundedness of the operator $\mathcal{A}$ follows from lemma 4.4 and boundedness of operators $C_\pm$, function V. Now we can provide the following improvement of Theorem 4.2.

Theorem 4.3. For each fixed $x > 0$ $\Phi(x,\cdot)$ is a unique solution in $\mathcal{H}$ of the equation $\mathcal{A}\phi = G$. Moreover, operator $\mathcal{A}$ has a bounded inverse operator $\mathcal{A}^{-1} = \mathcal{B}$, where:

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_{rr} & \mathcal{B}_{rd} \\ \mathcal{B}_{dr} & \mathcal{B}_{dd} \end{bmatrix},$$

$$\mathcal{B}_{rr}f = C_+(f\tilde{P}_+)P_+ - C_-(f\tilde{P}_+)P_-, \ (\mathcal{B}_{rd}f)(\rho) = \sum_{\mu\in\check{Z}_k} f(\mu)d_1(\rho,\mu)(P_+(\rho) - P_-(\rho)), \ \rho \in \mathbb{R},$$

$$(\mathcal{B}_{dr}f)(\rho) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(\xi)\tilde{P}_+(\xi)d_2(\rho,\xi)\hat{v}(\rho)d\xi, \ \rho \in \check{Z}_k, \ (\mathcal{B}_{dd}f)(\rho) = f(\rho) - \sum_{\mu\in\check{Z}_k} f(\mu)A(x,\rho,\mu), \ \rho \in \check{Z}_k.$$

Here and below $\tilde{P} := P^{-1}$ while P denotes as above the spectral mappings matrix.

Lemma 4.5. The following relation holds for nonreal $\rho \neq \mu$, $\rho,\mu \in \mathbb{C}\setminus\check{Z}_k$:

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\xi}{(\rho-\xi)(\xi-\mu)}(P_+(\xi) - P_-(\xi)) = \frac{1}{\rho-\mu}P(\mu) - \frac{1}{\rho-\mu}P(\rho) + \sum_{\xi\in\check{Z}_k} d_2(\xi,\mu)\hat{v}(\xi)\tilde{d}_1(\rho,\xi).$$

For $\rho \in \mathbb{C}\setminus(\mathbb{R}\cup\check{Z}_k), \mu \in \check{Z}_k$ one has:

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\xi}{\rho-\xi}d_1(\xi,\mu)(P_+(\xi) - P_-(\xi)) = \tilde{d}_1(\rho,\mu) - d_1(\rho,\mu)P(\rho) - \sum_{\xi\in\check{Z}_k} A(\xi,\mu)\tilde{d}_1(\rho,\xi).$$

For $\mu \in \mathbb{C}\setminus(\mathbb{R}\cup\check{Z}_k), \rho \in \check{Z}_k$ one has:

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\xi}{\xi-\mu}(P_+(\xi) - P_-(\xi))\tilde{d}_2(\rho,\xi)\hat{v}(\rho) = P(\mu)\tilde{d}_2(\rho,\mu)\hat{v}(\rho) - d_2(\rho,\mu)\hat{v}(\rho) + \sum_{\xi\in\check{Z}_k} d_2(\xi,\mu)\hat{v}(\xi)\tilde{A}(\rho,\xi).$$

For $\rho,\mu \in \check{Z}_k$ one has:

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} d_1(\xi,\mu)(P_+(\xi) - P_-(\xi))\tilde{d}_2(\rho,\xi)\hat{v}(\rho)d\xi = \tilde{A}(\rho,\mu) + A(\rho,\mu) - \sum_{\xi\in\check{Z}_k} A(\xi,\mu)\tilde{A}(\rho,\xi).$$

Symmetrically, one can obtain the following relations:

$$\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{d\xi}{(\rho-\xi)(\xi-\mu)}(\tilde{P}_+(\xi)-\tilde{P}_-(\xi))=\frac{1}{\rho-\mu}\tilde{P}(\mu)-\frac{1}{\rho-\mu}\tilde{P}(\rho)-\sum_{\xi\in\check{Z}_k}\tilde{d}_2(\xi,\mu)\hat{v}(\xi)d_1(\rho,\xi)$$

for $\rho,\mu\in\mathbb{C}\setminus(\mathbb{R}\cup\check{Z}_k)$,

$$\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{d\xi}{\rho-\xi}\tilde{d}_1(\xi,\mu)(\tilde{P}_+(\xi)-\tilde{P}_-(\xi))=d_1(\rho,\mu)-\tilde{d}_1(\rho,\mu)\tilde{P}(\rho)-\sum_{\xi\in\check{Z}_k}\tilde{A}(\xi,\mu)d_1(\rho,\xi)$$

for $\rho\in\mathbb{C}\setminus(\mathbb{R}\cup\check{Z}_k),\mu\in\check{Z}_k$,

$$\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{d\xi}{\xi-\mu}(\tilde{P}_+(\xi)-\tilde{P}_-(\xi))d_2(\rho,\xi)\hat{v}(\rho)=\tilde{P}(\mu)d_2(\rho,\mu)\hat{v}(\rho)-\tilde{d}_2(\rho,\mu)\hat{v}(\rho)+\sum_{\xi\in\check{Z}_k}\tilde{d}_2(\xi,\mu)\hat{v}(\xi)A(\rho,\xi)$$

for $\mu\in\mathbb{C}\setminus(\mathbb{R}\cup\check{Z}_k),\rho\in\check{Z}_k$,

$$-\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\tilde{d}_1(\xi,\mu)(\tilde{P}_+(\xi)-\tilde{P}_-(\xi))d_2(\rho,\xi)\hat{v}(\rho)d\xi=\tilde{A}(\rho,\mu)+A(\rho,\mu)-\sum_{\xi\in\check{Z}_k}\tilde{A}(\xi,\mu)A(\rho,\xi)$$

for $\rho,\mu\in\check{Z}_k$.

Proof. All the relations can be obtained in a similar way based on the relation:

$$\lim_{R\to\infty}\int\limits_{|\xi|=R}\frac{d\xi}{(\rho-\xi)(\xi-\mu)}P(\xi)=0,\ \ \rho,\mu\in\mathbb{C}\setminus(\mathbb{R}\cup\check{Z}_k),\tag{4.16}$$

that follows from lemma 4.1.

Let us show how to get, for instance, the forth relation. Multiplying (4.16) by $\Psi^{-1}(\mu)$ from the left, by $\tilde{\Psi}(\rho)$ from the right and applying the residue theorem we obtain:

$$\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{d\xi}{(\rho-\xi)(\xi-\mu)}\Psi^{-1}(\mu)(P_+(\xi)-P_-(\xi))\tilde{\Psi}(\rho)=$$

$$\sum_{\xi\in\check{Z}_k}\frac{1}{(\rho-\xi)(\xi-\mu)}\Psi^{-1}(\mu)P_{\langle -1\rangle}(\xi)\tilde{\Psi}(\rho)+\frac{1}{\rho-\mu}\tilde{\Psi}^{-1}(\mu)\tilde{\Psi}(\rho)-\frac{1}{\rho-\mu}\Psi^{-1}(\mu)\Psi(\rho),\ \ \rho,\mu\in\mathbb{C}\setminus(\mathbb{R}\cup\check{Z}_k).$$

Using lemma 4.3 we rewrite this as follows:

$$\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{d\xi}{(\rho-\xi)(\xi-\mu)}\Psi^{-1}(\mu)(P_+(\xi)-P_-(\xi))\tilde{\Psi}(\rho)=$$

$$\sum_{\xi\in\check{Z}_k}D(\xi,\mu)\hat{v}(\xi)\tilde{D}(\rho,\xi)+\tilde{D}(\rho,\mu)-D(\rho,\mu),\ \ \rho,\mu\in\mathbb{C}\setminus(\mathbb{R}\cup\check{Z}_k).$$

Taking (for arbitrary fixed $\rho\in\mathbb{C}\setminus(\mathbb{R}\cup\check{Z}_k)$, $\mu_0\in\check{Z}_k$) the coefficients $[...]_{\langle 0\rangle}\big|_{\mu=\mu_0}$ in Laurent series of both sides we arrive at:

$$\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{d\xi}{\rho-\xi}d_1(\xi,\mu_0)(P_+(\xi)-P_-(\xi))\tilde{\Psi}(\rho)=$$

$$-\sum_{\xi\in\check{Z}_k}A(\xi,\mu_0)\tilde{D}(\rho,\xi)+\tilde{D}(\rho,\mu_0)-D(\rho,\mu_0),\ \ \mu_0\in\check{Z}_k,\rho\in\mathbb{C}\setminus(\mathbb{R}\cup\check{Z}_k).$$

Finally, taking (for arbitrary $\rho_0, \mu_0 \in \check{Z}_k$) the coefficients $[...]_{\langle 0 \rangle}\big|_{\rho=\rho_0}$ in Laurent series of both sides and multiplying them by $\hat{v}(\rho_0)$ we obtain the required relation. $\square$

Proof of Theorem 4.3. Boundedness of the operator $\mathcal{B}$ follows from lemma 4.4 and boundedness of operators $C_\pm$, functions $V, P_\pm, \tilde{P}_\pm$. The proof consists in the direct calculations of elements of the operator matrices $\mathcal{AB}$ and $\mathcal{BA}$.

Position (r,r). Let $\varphi = \mathcal{B}_{rr} f$. We rewrite it as $\varphi(\rho) = f(\rho) + \left[C_-(f\tilde{P}_+)\right](\rho)(P_+(\rho) - P_-(\rho))$. Then $(C\varphi)(\rho)$ for arbitrary fixed nonreal $\rho \in \mathbb{C} \setminus \check{Z}_k$ can be written in the following form:

$$(C\varphi)(\rho) = (Cf)(\rho) - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[C_-(f\tilde{P}_+)\right](\mu) F_\rho(\mu) d\mu,$$

where $F_\rho(\mu) := (\rho - \mu)^{-1}(P_+(\mu) - P_-(\mu))$. Using the relation

$$-\int_{-\infty}^{\infty} (C_- F_1)(\mu) F_2(\mu) d\mu = \int_{-\infty}^{\infty} F_1(\mu)(C_+ F_2)(\mu) d\mu$$

we rewrite it as follows:

$$(C\varphi)(\rho) = (Cf)(\rho) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} (f\tilde{P}_+)(\mu)(C_+ F_\rho)(\mu) d\mu.$$

Now let us consider $(CF_\rho)(\mu)$. For nonreal $\mu \in \mathbb{C} \setminus \check{Z}_k$ using lemma 4.5 we calculate:

$$(CF_\rho)(\mu) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\xi}{(\rho - \xi)(\xi - \mu)} (P_+(\xi) - P_-(\xi)) = \frac{1}{\rho - \mu} P(\mu) - \frac{1}{\rho - \mu} P(\rho) + \sum_{\xi \in \check{Z}_k} d_2(\xi, \mu) \hat{v}(\xi) \tilde{d}_1(\rho, \xi).$$

Taking the limit as $\mathrm{Im}\mu \to +0$ (while $\rho \in \mathbb{C} \setminus (\mathbb{R} \cup \check{Z}_k)$ remains fixed) we obtain:

$$(C_+ F_\rho)(\mu) = \frac{1}{\rho - \mu} P_+(\mu) - \frac{1}{\rho - \mu} P(\rho) + \sum_{\xi \in \check{Z}_k} d_2(\xi, \mu) \hat{v}(\xi) \tilde{d}_1(\rho, \xi).$$

Thus, we can write:

$$(C\varphi)(\rho) = (Cf)(\rho) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} (f\tilde{P}_+)(\mu) \left\{ \frac{1}{\rho - \mu} P_+(\mu) - \frac{1}{\rho - \mu} P(\rho) + \sum_{\xi \in \check{Z}_k} d_2(\xi, \mu) \hat{v}(\xi) \tilde{d}_1(\rho, \xi) \right\} d\mu =$$

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} (f\tilde{P}_+)(\mu) \left\{ \sum_{\xi \in \check{Z}_k} d_2(\xi, \mu) \hat{v}(\xi) \tilde{d}_1(\rho, \xi) \right\} d\mu + \left[C(f\tilde{P}_+)\right](\rho) P(\rho)$$

that yields the following relation:

$$(\mathcal{A}_{rr}\mathcal{B}_{rr} f)(\rho) = (\mathcal{A}_{rr}\varphi)(\rho) = \left[C_+(f\tilde{P}_+)\right](\rho) P_+(\rho) - \left[C_-(f\tilde{P}_+)\right](\rho) P_-(\rho) V(\rho) +$$

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} (f\tilde{P}_+)(\mu) \left\{ \sum_{\xi \in \check{Z}_k} d_2(\xi, \mu) \hat{v}(\xi) \tilde{d}_1(\rho, \xi) \right\} (I - V(\rho)) d\mu.$$

On the other hand we have:

$$(\mathcal{A}_{rd}\mathcal{B}_{dr} f)(\rho) = \sum_{\xi \in \check{Z}_k} \left\{ -\frac{1}{2\pi i} \int_{-\infty}^{\infty} f(\mu) \tilde{P}_+(\mu) d_2(\xi, \mu) \hat{v}(\xi) d\mu \right\} \tilde{d}_1(\rho, \xi)(I - V(\rho))$$

and thus arrive at:

$$(\mathcal{A}_{rr}\mathcal{B}_{rr}f)(\rho)+(\mathcal{A}_{rd}\mathcal{B}_{dr}f)(\rho)=\Big[C_+(f\tilde{P}_+)\Big](\rho)P_+(\rho)-\Big[C_-(f\tilde{P}_+)\Big](\rho)P_-(\rho)V(\rho).$$

Taking into account that $P_+=P_-V$ and $\tilde{P}=P^{-1}$ we obtain finally $(\mathcal{A}_{rr}\mathcal{B}_{rr}f)(\rho)+(\mathcal{A}_{rd}\mathcal{B}_{dr}f)(\rho)=[(C_+-C_-)(f\tilde{P}_+)](\rho)P_+(\rho)=f(\rho)$.

Now let $f=\mathcal{A}_{rr}\varphi$. Then $(f\tilde{P}_+)(\rho)=(\varphi\tilde{P}_+)(\rho)+(C_-\varphi)(\rho)(\tilde{P}_+(\rho)-\tilde{P}_-(\rho))$. Consider the function $[C(f\tilde{P}_+)](\rho)$. Proceeding as above we obtain:

$$[C(f\tilde{P}_+)](\rho)=[C(\varphi\tilde{P}_+)](\rho)+\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\varphi(\mu)[C_+\tilde{F}_\rho](\mu)d\mu,$$

where $\tilde{F}_\rho(\mu):=(\rho-\mu)^{-1}(\tilde{P}_+(\mu)-\tilde{P}_-(\mu))$. Using lemma 4.5 we calculate:

$$[C_+\tilde{F}_\rho](\mu)=\frac{1}{\rho-\mu}\tilde{P}_+(\mu)-\frac{1}{\rho-\mu}\tilde{P}(\rho)-\sum_{\xi\in\check{Z}_k}\tilde{d}_2(\xi,\mu)\hat{v}(\xi)d_1(\rho,\xi),$$

$$[C(f\tilde{P}_+)](\rho)P(\rho)=(C\varphi)(\rho)-\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\varphi(\mu)\left\{\sum_{\xi\in\check{Z}_k}\tilde{d}_2(\xi,\mu)\hat{v}(\xi)d_1(\rho,\xi)\right\}P(\rho)d\mu$$

that yields finally:

$$(\mathcal{B}_{rr}\mathcal{A}_{rr}\varphi)(\rho)=(\mathcal{B}_{rr}f)(\rho)=\varphi(\rho)-\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\varphi(\mu)\left\{\sum_{\xi\in\check{Z}_k}\tilde{d}_2(\xi,\mu)\hat{v}(\xi)d_1(\rho,\xi)\right\}(P_+(\rho)-P_-(\rho))d\mu.$$

On the other hand we have:

$$(\mathcal{B}_{rd}\mathcal{A}_{dr}\varphi)(\rho)=\sum_{\xi\in\check{Z}_k}\left\{\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\varphi(\mu)\tilde{d}_2(\xi,\mu)\hat{v}(\xi)d\mu\right\}d_1(\rho,\xi)(P_+(\rho)-P_-(\rho))$$

and thus $(\mathcal{B}_{rr}\mathcal{A}_{rr}\varphi)(\rho)+(\mathcal{B}_{rd}\mathcal{A}_{dr}\varphi)(\rho)=\varphi(\rho)$.

Position (r,d). Calculate $\mathcal{A}_{rr}\mathcal{B}_{rd}+\mathcal{A}_{rd}\mathcal{B}_{dd}$. Let $\varphi=\mathcal{B}_{rd}f$. Then

$$\mathcal{A}_{rr}\varphi(\rho)=\sum_{\mu\in\check{Z}_k}f(\mu)\left[C_+R_\mu-(C_-R_\mu)V\right](\rho),$$

where $R_\mu(\rho):=d_1(\rho,\mu)(P_+(\rho)-P_-(\rho))$. Consider for nonreal $\rho\in\mathbb{C}\setminus\check{Z}_k$ the function:

$$CR_\mu(\rho)=-\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{d\xi}{\rho-\xi}d_1(\xi,\mu)(P_+(\xi)-P_-(\xi)).$$

Using lemma 4.5 we rewrite this as follows:

$$CR_\mu(\rho)=-\tilde{d}_1(\rho,\mu)+d_1(\rho,\mu)P(\rho)+\sum_{\xi\in\check{Z}_k}A(\xi,\mu)\tilde{d}_1(\rho,\xi).$$

Taking the limits $\pm\mathrm{Im}\rho\to 0$ we obtain:

$$\left[C_+R_\mu-(C_-R_\mu)V\right](\rho)=-\tilde{d}_1(\rho,\mu)(I-V(\rho))+d_1(\rho,\mu)(P_+(\rho)-P_-(\rho)V(\rho))$$
$$+\sum_{\xi\in\check{Z}_k}A(\xi,\mu)\tilde{d}_1(\rho,\xi)(I-V(\rho)).$$

Thus, taking into account that $P_+=P_-V$ we arrive at:

$$(\mathcal{A}_{rr}\mathcal{B}_{rd})(\rho)=\left\{\sum_{\mu\in\check{Z}_k}f(\mu)F(\rho,\mu)\right\}(I-V(\rho)),$$

where

$$F(\rho,\mu)=\sum_{\xi\in\check{Z}_k}A(\xi,\mu)\tilde{d}_1(\rho,\xi)-\tilde{d}_1(\rho,\mu).$$

On the other hand direct calculation yields:

$$(\mathcal{A}_{rd}\mathcal{B}_{dd}f)(\rho)=\left\{\sum_{\xi\in\check{Z}_k}\left(f(\xi)-\sum_{\mu\in\check{Z}_k}f(\mu)A(\xi,\mu)\tilde{d}_1(\rho,\xi)\right)\right\}(I-V(\rho)),$$

that can be rewritten as

$$(\mathcal{A}_{rd}\mathcal{B}_{dd})(\rho)=-\left\{\sum_{\mu\in\check{Z}_k}f(\mu)F(\rho,\mu)\right\}(I-V(\rho)).$$

Thus, we have $\mathcal{A}_{rr}\mathcal{B}_{rd}+\mathcal{A}_{rd}\mathcal{B}_{dd}=0$. Symmetrical calculations show that $\mathcal{B}_{rr}\mathcal{A}_{rd}+\mathcal{B}_{rd}\mathcal{A}_{dd}=0$.

Position (d,r). Calculate $\mathcal{A}_{dr}\mathcal{B}_{rr}+\mathcal{A}_{dd}\mathcal{B}_{dr}$. Let $\varphi=\mathcal{B}_{dr}f$. Then $\mathcal{A}_{dd}\varphi(\rho)$ can be written in the following form:

$$\mathcal{A}_{dd}\varphi(\rho)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(\mu)\tilde{P}_+(\mu)\left[-d_2(\rho,\mu)\hat{v}(\rho)+\sum_{\xi\in\check{Z}_k}d_2(\xi,\mu)\hat{v}(\xi)\tilde{A}(\rho,\xi)\right]d\mu.$$

Consider the function $R_\rho(\mu):=(P_+(\mu)-P_-(\mu))\tilde{d}_2(\rho,\mu)\hat{v}(\rho)$ (where $\rho\in\check{Z}_k$ considered as a parameter). By virtue of lemma 4.5 one has for nonreal $\mu\in\mathbb{C}\setminus(\mathbb{R}\cup\check{Z}_k)$:

$$(CR_\rho)(\mu)=P(\mu)\tilde{d}_2(\rho,\mu)\hat{v}(\rho)-d_2(\rho,\mu)\hat{v}(\rho)+\sum_{\xi\in\check{Z}_k}d_2(\xi,\mu)\hat{v}(\xi)\tilde{A}(\rho,\xi).$$

Taking the limit as $\mathrm{Im}\mu\to+0$ we obtain for real μ:

$$(C_+R_\rho)(\mu)=P_+(\mu)\tilde{d}_2(\rho,\mu)\hat{v}(\rho)-d_2(\rho,\mu)\hat{v}(\rho)+\sum_{\xi\in\check{Z}_k}d_2(\xi,\mu)\hat{v}(\xi)\tilde{A}(\rho,\xi).$$

Thus, we can write:

$$\mathcal{A}_{dd}\varphi(\rho)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(\mu)\tilde{P}_+(\mu)\left[(C_+R_\rho)(\mu)-P_+(\mu)\tilde{d}_2(\rho,\mu)\hat{v}(\rho)\right]d\mu=$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(\mu)\tilde{P}_+(\mu)(C_+R_\rho)(\mu)d\mu-\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(\mu)\tilde{d}_2(\rho,\mu)\hat{v}(\rho)d\mu.$$

Using the relation

$$\int_{-\infty}^{\infty}F_1(\mu)(C_+F_2)(\mu)d\mu=-\int_{-\infty}^{\infty}(C_-F_1)(\mu)F_2(\mu)d\mu$$

we rewrite this as follows:

$$(\mathcal{A}_{dd}\mathcal{B}_{dr}f)\rho=\mathcal{A}_{dd}\varphi(\rho)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[C_-(f\tilde{P}_+)\right](\mu)R_\rho(\mu)d\mu-\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(\mu)\tilde{d}_2(\rho,\mu)\hat{v}(\rho)d\mu=$$

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[C_-(f\tilde{P}_+)\right](\mu)(P_+(\mu)-P_-(\mu))\tilde{d}_2(\rho,\mu)\hat{v}(\rho)d\mu-\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(\mu)\tilde{d}_2(\rho,\mu)\hat{v}(\rho)d\mu.$$

Now let $\varphi=\mathcal{B}_{rr}f$. We rewrite it as $\varphi(\mu)=f(\mu)+C_-\left[(f\tilde{P}_+)\right](\mu)(P_+(\mu)-P_-(\mu))$. Then

$$(\mathcal{A}_{dr}\mathcal{B}_{rr}f)(\rho)=(\mathcal{A}_{dr}\varphi)(\rho)=$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(\mu)\tilde{d}_2(\rho,\mu)\hat{v}(\rho)d\mu + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[C_-(f\tilde{P}_+)\right](\mu)(P_+(\mu)-P_-(\mu))\tilde{d}_2(\rho,\mu)\hat{v}(\rho)d\mu.$$

Analogous calculations yield

$$(\mathcal{B}_{dd}\mathcal{A}_{dr}\varphi)(\rho) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[C_-\varphi\right](\mu)(\tilde{P}_+(\mu)-\tilde{P}_-(\mu))d_2(\rho,\mu)\hat{v}(\rho)d\mu + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\varphi(\mu)\tilde{P}_+(\mu)d_2(\rho,\mu)\hat{v}(\rho)d\mu.$$

On the other hand, if $f = \mathcal{A}_{rr}\varphi$ then $(f\tilde{P}_+)(\mu) = \varphi(\mu)\tilde{P}_+(\mu) + [C_-\varphi](\mu)(\tilde{P}_+(\mu)-\tilde{P}_-(\mu))$ and

$$(\mathcal{B}_{dr}\mathcal{A}_{rr}\varphi)(\rho) = (\mathcal{B}_{dr}f)(\rho) =$$

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\varphi(\mu)\tilde{P}_+(\mu)d_2(\rho,\mu)\hat{v}(\rho)d\mu - \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[C_-\varphi\right](\mu)(\tilde{P}_+(\mu)-\tilde{P}_-(\mu))d_2(\rho,\mu)\hat{v}(\rho)d\mu.$$

Thus, we have $\mathcal{B}_{dd}\mathcal{A}_{dr} + \mathcal{B}_{dr}\mathcal{A}_{rr} = 0$.

Position (d,d). Let $\varphi = \mathcal{B}_{rd}f$. Then

$$(\mathcal{A}_{dr}\varphi)(\rho) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left\{\sum_{\mu\in\check{Z}_k} f(\mu)d_1(\xi,\mu)\right\}(P_+(\xi)-P_-(\xi))\tilde{d}_2(\rho,\xi)\hat{v}(\rho)d\xi.$$

Using lemma 4.5 we can rewrite it as follows

$$(\mathcal{A}_{dr}\mathcal{B}_{rd}f)(\rho) = (\mathcal{A}_{dr}\varphi)(\rho) = \sum_{\mu\in\check{Z}_k} f(\mu)\left\{\tilde{A}(\rho,\mu) + A(\rho,\mu) - \sum_{\xi\in\check{Z}_k} A(\xi,\mu)\tilde{A}(\rho,\xi)\right\}.$$

On the other hand we have:

$$(\mathcal{A}_{dd}\mathcal{B}_{dd}f)(\rho) = f(\rho) - \sum_{\mu\in\check{Z}_k} f(\mu)A(\rho,\mu) - \sum_{\xi\in\check{Z}_k}\left\{f(\xi) - \sum_{\mu\in\check{Z}_k} f(\mu)A(\xi,\mu)\right\}\tilde{A}(\rho,\xi)$$

and thus we obtain $(\mathcal{A}_{dr}\mathcal{B}_{rd}f)(\rho) + (\mathcal{A}_{dd}\mathcal{B}_{dd}f)(\rho) = f(\rho)$. Symmetrical calculations yield $\mathcal{B}_{dr}\mathcal{A}_{rd} + \mathcal{B}_{dd}\mathcal{A}_{dd} = E$. □

Now we can formulate the constructive procedure for solution of the problem $IP(k)$.

Procedure 4.1. Given scattering data J_k (with some fixed $k \in \{1,\dots,p\}$) and $\nu_{0j}, \sigma_j, \sigma_{j1}, \sigma_{j2}$, $j = \overline{1,p}$.

1. Choose $\tilde{L}$ with the same $\nu_{0j}, \sigma_j, \sigma_{j1}, \sigma_{j2}$, $j = \overline{1,p}$ satisfying the conditions G, R_0, R_∞.
2. Calculate $v(\rho)$, $\rho \in \mathbb{R}\cup\check{Z}_k$ using (4.10) and lemma 4.2.
3. From given $v(\rho)$, $\rho \in \mathbb{R}\cup\check{Z}_k$ and $\tilde{L}$ find $V(x,\rho)$, $\tilde{d}_j(x,\rho,\mu), j = 1,2$, $\tilde{A}(x,\rho,\mu)$.
4. For each fixed $x > 0$ find $\Phi(x,\rho), \rho \in \mathbb{R}\cup\check{Z}_k$ as a (unique) solution of the linear system (4.13), (4.15).
5. Given $\Phi(x,\rho)$ calculate $P_1(x,\rho)$, $x > 0$, $\rho \in \mathbb{C}\setminus\mathbb{R}\cup\check{Z}_k$ via (4.12).
6. Given $P_1(x,\rho)$ find $f_k(x,\rho) = P_{11}(x,\rho)\tilde{f}_k(x,\rho) + P_{11}(x,\rho)\tilde{f}'_k(x,\rho)$, $\psi_{kk}(x,\rho) = P_{11}(x,\rho)\tilde{\psi}_{kk} + P_{11}(x,\rho)\tilde{\psi}'_{kk}(x,\rho)$.
7. Calculate $q_k(x) = f''_k(x,\rho)/f_k(x,\rho) + \rho^2 - \nu_{0k}x^{-2}$ (where ρ for each fixed $x > 0$ is arbitrary such that $f_k(x,\rho) \neq 0$).

5 Inverse scattering on the graph

Here we consider the following "complete" inverse scattering problem.

Problem $IP(\Gamma)$**.** Given scattering data J $(= \{J_k\}_{k=1}^{p-1})$ find the potential on Γ, i.e. all the functions $q_k, k = \overline{1,p}$.

First we note that the procedure described in previous section allows us to recover (uniquely) the potentials $q_k, k = \overline{1,p-1}$. Thus, in order to complete our solution of the problem we have to recover q_p. Let us

consider the matching conditions for Weyl-type solution $\psi_1(\rho)$. The following relation is a direct sequence of (3.2):

$$\sum_{j=1}^{p} \frac{U_{j2}(\psi_{1j}(\cdot,\rho))}{U_{j1}(\psi_{1j}(\cdot,\rho))} = 0. \tag{5.1}$$

Since for $j = \overline{2,p}$ $\psi_{kj}(x,\rho) = \gamma_{kj}(\rho) f_j(x,\rho)$, we have:

$$\frac{U_{j2}(\psi_{1j}(\cdot,\rho))}{U_{j1}(\psi_{1j}(\cdot,\rho))} = \frac{\sigma_{j1} + \sigma_{j2} m_j(\lambda)}{\sigma_j},$$

where $m_j(\lambda) = b_{j2}(\rho)/b_{j1}(\rho)$ are the local Weyl functions on the rays $\mathcal{R}_j$. Thus, (5.1) can be rewritten as follows:

$$\frac{\sigma_{p1} + \sigma_{p2} m_p(\lambda)}{\sigma_p} = -\sum_{j=2}^{p-1} \frac{\sigma_{j1} + \sigma_{j2} m_j(\lambda)}{\sigma_j} - \frac{U_{12}(\psi_{11}(\cdot,\rho))}{U_{11}(\psi_{11}(\cdot,\rho))}. \tag{5.2}$$

Now we note that all the Weyl functions $m_j, j = \overline{2,p-1}$ are known since corresponding potentials q_j were already recovered from the solution of problems $IP(j), j = \overline{2,p-1}$. Moreover, the Weyl-type solution $\psi_{11}(x,\rho)$ can also be found from the solution of the problem $IP(1)$. Thus, all the terms in the right-hand side of (5.2) are known and we can use (5.2) to find the Weyl function m_p that actually completes the solution of the problem $IP(\Gamma)$.

The following theorem summarizes our results.

Theorem 5.1. Specification of the scattering data J determines uniquely the potential on Γ. The functions $q_k, k = \overline{1,p}$ can be recovered by the following procedure.
1. For $k = \overline{1,p-1}$ using the procedure 4.1 solve the problems $IP(k)$ and recover q_k. Calculate $\psi_{11}(x,\rho)$ (while solving the problem $IP(1)$).
2. Find m_p from (5.2).
3. Given m_p recover $q_p(x), x > 0$ by solving the (local) inverse spectral problem on the semi-axis $x > 0$ by means of the procedure described in [27].

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